On Complete Sets of Polarization Observables *

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A new criterion is developed which provides a check as to whether a chosen set of polarization observables is complete with respect to the determination of all independent T-matrix elements of a reaction of the type $a + b \to c + d + \cdots$. As an illustrative example, this criterion is applied to the longitudinal observables of deuteron electrodisintegration.

I. INTRODUCTION

The major reason for studying polarization phenomena in reactions of the type $a + b \to c + d + \cdots$ lies in the fact that only the use of polarization degrees of freedom allows one to obtain complete information on all possible reaction matrix elements. Without polarization, the cross section is given by the incoherent sum of squares of the reaction matrix elements only. Thus small amplitudes are masked by the dominant ones. On the other hand, small amplitudes very often contain interesting information on subtle dynamical effects. This is the place where polarization observables enter, because such observables in general contain interference terms of the various matrix elements in different ways. Thus a small amplitude may be considerably amplified by the interference with a dominant matrix element. An example is provided by the influence of the small electric form factor of the neutron on the transverse in-plane component of the neutron polarization in quasi-free deuteron electrodisintegration [1–3]. It is just this feature for which polarization physics has become such an important topic in various branches of physics.

In view of the great number of possible polarization observables, i.e., beam and target asymmetries and/or polarization components of various outgoing particles, it is natural to ask which set of observables allows in principle a complete determination of all reaction matrix elements. Such sets are called complete sets of observables for obvious reasons. The usual strategies follow the explicit construction of a particular complete set. But this is practical only for a small number of matrix elements as in pion photoproduction [4,5]. With an increasing number of matrix elements as in deuteron photo- and electrodisintegration [6–8], for example, this method becomes more and more complicated. A general necessity statement as to the selection of polarization observables has been proven by Simonius [9], which excludes certain subsets from being complete. However, this statement is not specific enough. For this reason we want to study the alternative question: Given an appropriate subset of all observables, is there a criterion which allows one to decide whether this set constitutes a complete set of observables.

First we consider in Sect. 2 a simple example. The essential idea is then developed in Sect. 3 for the case of a general n-dimensional real quadratic form. In Sect. 4 we apply this criterion to the observables of a general reaction $a + b \to c + d + \cdots$ by rewriting the hermitean forms of the observables into real quadratic forms in the real and imaginary parts of the T-matrix elements. An illustrative example is considered for the longitudinal observables of deuteron electrodisintegration in Sect. 5.

II. A SIMPLE CASE FOR ILLUSTRATION

We will begin by considering a reaction $a + b \to c + d + \cdots$ of particles having spins s_a, s_b, $s_c, \ldots$ where angular momentum conservation requires that both $s_a + s_b$ and $s_c + s_d + \cdots$ are either integer or half-integer. Then the number of T-matrix elements for this reaction is given by $N = N(s_a)N(s_b)N(s_c) \cdots$ where for massive particles one has

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$$N(s_\alpha) = 2s_\alpha + 1\,, \tag{1}$$

while for massless particles

$$N(s_\alpha) = 2\,. \tag{2}$$

In case parity conservation holds, the number of independent T-matrix elements reduces to $n = N/2$ or $n = (N-1)/2$, depending on whether N is even or odd. Since any observable is given as a hermitean form in the T-matrix elements, the number of linearly independent observables is n^2. This does, however, not mean, that these observables are independent of each other in a more general sense, because on can find quadratic relations between them. In fact, since each T-matrix element is in general a complex number, but one overall phase is undetermined, one has the complete information on the reaction contained in a set of $2n - 1$ real numbers. It is therefore argued that a set of $2n-1$ properly chosen observables should suffice to determine completely all T-matrix elements. However, the solution is in general not unique but contains discrete ambiguities due to the fact that the solution implies the determination of the roots of a higher order polynomial, and therefore one needs additional information from further observables, like relative magnitudes etc. [4,5].

We will illustrate this feature by considering the simplest nontrivial case of two independent complex matrix elements T_1 and T_2. It can be realized by the absorption of a scalar particle by a $s = 1/2$ particle leading to a $s = 1/2$ particle. In this case any observable is represented by a 2×2 matrix for which we may choose besides the unit matrix (σ_0) the three Pauli matrices (σ_i, $i = 1, 2, 3$). Thus we have a set of four linearly independent observables

$$\mathcal{O}_\alpha = T^\dagger \sigma_\alpha T\,, \quad (\alpha = 0, \ldots, 3)\,, \tag{3}$$

or in greater detail

$$\mathcal{O}_0 = |T_1|^2 + |T_2|^2\,, \tag{4}$$
$$\mathcal{O}_1 = 2\,\Re e(T_1^* T_2)\,, \tag{5}$$
$$\mathcal{O}_2 = 2\,\Im m(T_1^* T_2)\,, \tag{6}$$
$$\mathcal{O}_3 = |T_1|^2 - |T_2|^2\,. \tag{7}$$

Evidently, the following relations hold

$$|T_1|^2 |T_2|^2 = \frac{1}{4}(\mathcal{O}_1^2 + \mathcal{O}_2^2)\,, \tag{8}$$
$$\mathcal{O}_3^2 = \mathcal{O}_0^2 - 4|T_1|^2 |T_2|^2$$
$$= \mathcal{O}_0^2 - (\mathcal{O}_1^2 + \mathcal{O}_2^2)\,, \tag{9}$$

and therefore

$$\mathcal{O}_0^2 \geq \mathcal{O}_1^2 + \mathcal{O}_2^2\,. \tag{10}$$

The relation in (9) constitutes just the above mentioned quadratic relation between observables for this case.

Without loss of generality we can choose the undetermined overall phase such that T_1 becomes real and is > 0. Then one can determine T_1 and T_2 from three observables, for example from $\mathcal{O}_0$, $\mathcal{O}_1$ and $\mathcal{O}_2$. Explicitly one finds

$$T_1^2 = \frac{1}{2}\left(\mathcal{O}_0 \pm \sqrt{\mathcal{O}_0^2 - (\mathcal{O}_1^2 + \mathcal{O}_2^2)}\right)\,, \tag{11}$$
$$T_2 = \frac{1}{2T_1}(\mathcal{O}_1 + i\mathcal{O}_2)\,. \tag{12}$$

Because of the relation (10), the right hand side of (11) is nonnegative and, therefore, the quadratic equation (11) for T_1 yields in general two real solutions. This is the discrete ambiguity mentioned in the introduction. In order to determine the proper solution, one has to consider the observable $\mathcal{O}_3$. Inserting the general solution, one finds

$$\mathcal{O}_3 = \pm\sqrt{\mathcal{O}_0^2 - (\mathcal{O}_1^2 + \mathcal{O}_2^2)}\,, \tag{13}$$

which corresponds to the relation (9). Thus the sign of $\mathcal{O}_3$ selects in this case the proper solution.

An alternative way to determine the T-matrix elements from the observables is to express all bilinear products of the form $T_i^* T_j$ by linear expressions in the observables [7], which reads in the considered case

$$T_1^* T_1 = \frac{1}{2} (\mathcal{O}_0 + \mathcal{O}_3) \,, \tag{14}$$

$$T_1^* T_2 = \frac{1}{2} (\mathcal{O}_1 + i\mathcal{O}_2) \,, \tag{15}$$

$$T_2^* T_2 = \frac{1}{2} (\mathcal{O}_0 - \mathcal{O}_3) \,. \tag{16}$$

Assuming again T_1 real and > 0, one finds as solution

$$T_1 = \sqrt{\frac{1}{2} (\mathcal{O}_0 + \mathcal{O}_3)} \,, \tag{17}$$

$$T_2 = \frac{\mathcal{O}_1 + i\mathcal{O}_2}{\sqrt{2(\mathcal{O}_0 + \mathcal{O}_3)}} \,, \tag{18}$$

which coincides with the previous solution. Evidently, the quadratic relation between the observables is fulfilled by this explicit solution. In this case, one obtains directly the explicit form of the T-matrix elements expressed in terms of observables. The disadvantage of this approach is that one has no choice with respect to the observables to be measured because they are determined by the linear expressions of $T_i^* T_j$.

III. GENERAL CRITERION FOR A COMPLETE SET OF REAL QUADRATIC FORMS

In this section we will study the following question: Given an n-dimensional real vector $x = (x_1, \ldots, x_n)$ and n real quadratic forms

$$f^\alpha(x) = \frac{1}{2} \sum_{ij} x_i H_{ij}^\alpha x_j \,, \tag{19}$$

where $\alpha = 1, \ldots, n$, what is the criterion that the set of n equations

$$f^\alpha(x) = c^\alpha \tag{20}$$

for given constants c^α possesses a solution x^0 with allowance for possible discrete ambiguities? In other words, is the set of equations in (20) invertible? A sufficient condition for the inversion is that in the vicinity of x^0 the Jacobian $Df|_{x^0}$ is nonvanishing, i.e.

$$Df|_{x^0} = \left| \frac{\partial f^\alpha}{\partial x_j} \right| \neq 0 \,. \tag{21}$$

This means that the n vectors v^α defined by the partial derivatives

$$\frac{\partial f^\alpha}{\partial x_j} = H_{ji}^\alpha x_i^0 \,, \tag{22}$$

according to

$$v^\alpha = \begin{pmatrix} \frac{\partial f^\alpha}{\partial x_1} \\ \vdots \\ \frac{\partial f^\alpha}{\partial x_n} \end{pmatrix} = \begin{pmatrix} H_{1i}^\alpha x_i^0 \\ \vdots \\ H_{ni}^\alpha x_i^0 \end{pmatrix} \,, \tag{23}$$

are linearly independent. The question thus is: under which condition is this true for an arbitrary x^0? The n vectors v^α are linear forms in x^0

$$v^\alpha = \begin{pmatrix} H_{11}^\alpha \\ \vdots \\ H_{n1}^\alpha \end{pmatrix} x_1^0 + \cdots + \begin{pmatrix} H_{1n}^\alpha \\ \vdots \\ H_{nn}^\alpha \end{pmatrix} x_n^0 \,, \tag{24}$$

which we may write in the form

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$$v^\alpha = \sum_{k=1}^{n} w^\alpha(k)x_k^0 \,, \tag{25}$$

where we have introduced

$$w^\alpha(k) = \begin{pmatrix} H_{1k}^\alpha \\ \vdots \\ H_{nk}^\alpha \end{pmatrix}. \tag{26}$$

A necessary and sufficient condition for the linear independence of the n vectors v^α is that the determinant built from the n vectors v^α is nonvanishing

$$\det[v^1, \dots, v^n] \neq 0 \,, \tag{27}$$

which with the help of (25) is written in the equivalent form

$$\sum_{k_1=1}^{n} \cdots \sum_{k_n=1}^{n} x_{k_1}^0 \cdots x_{k_n}^0 \det[w^1(k_1), \dots, w^n(k_n)] \neq 0 \,. \tag{28}$$

Evidently, a necessary condition for the nonvanishing of this determinant is that of the various sets of n vectors

$$W(k_1, \dots, k_n) = \{w^\alpha(k_\alpha);\ \alpha = 1, \dots, n;\ k_\alpha \in \{1, \dots, n\}\} \tag{29}$$

at least one is a set of linearly independent vectors, i.e.,

$$\det[w^1(k_1), \dots, w^n(k_n)] = \begin{vmatrix} H_{1k_1}^1 & \cdots & H_{1k_n}^n \\ \vdots & & \vdots \\ H_{nk_1}^1 & \cdots & H_{nk_n}^n \end{vmatrix} \neq 0 \,. \tag{30}$$

Note that the k_α need not be different. If none of these determinants is nonvanishing, the n vectors v^α certainly cannot be linearly independent. Moreover, if this condition is fulfilled exactly for one set of $\{k_\alpha\}$ while the determinants of all other $W(k_1, \dots, k_n)$ vanish, then this condition is also sufficient. In case one finds $|W(k_1, \dots, k_n)| \neq 0$ for more than one set $\{k_\alpha\}$, then it could in principle happen that for a specific choice of x^0 the set v^α becomes linearly dependent, although it will in general be rather unlikely to happen. But one has to check this in a given situation. This is the criterion we were looking for and which now we will apply to the hermitean forms of a set of observables.

IV. COMPLETE SETS OF OBSERVABLES

We represent an observable by an $n \times n$ hermitean form f^α in the complex n-dimensional variable z

$$f^\alpha(z) = \frac{1}{2} \sum_{j'j} z_{j'}^* H_{j'j}^\alpha z_j \,, \tag{31}$$

where hermiticity requires

$$(H_{j'j}^\alpha)^* = H_{jj'}^\alpha \,. \tag{32}$$

Now we will rewrite this form into a real quadratic form by introducing

$$z = x + iy \,, \tag{33}$$
$$H^\alpha = A^\alpha + i\,B^\alpha \,, \tag{34}$$

where A^α and B^α are real matrices and A^α is symmetric whereas B^α is antisymmetric. Considering further the fact that one overall phase is arbitrary, we may choose $y_{j_0} = 0$ for one index j_0 and then obtain

$$f^\alpha(x + iy) = \frac{1}{2}\Big[\sum_{j'j} x_{j'} A_{j'j}^\alpha x_j + \sum_{\tilde{j}'\tilde{j}} y_{j'} A_{j'j}^\alpha y_j + 2 \sum_{\tilde{j}'j} y_{j'} B_{j'j}^\alpha x_j \Big] \,, \tag{35}$$

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where the tilde over a summation index indicates that the index j_0 has to be left out. Introducing now an m-dimensional real vector u ($m = 2n - 1$) by

$$u = (x_1, \ldots, x_n, y_1, \ldots, y_{j_0-1}, y_{j_0+1}, \ldots, y_n), \tag{36}$$

one can represent the $n \times n$ hermitean form by an $m \times m$ real quadratic form

$$\widetilde{f}^\alpha(u) = \frac{1}{2} \sum_{l'l=1}^{m} u_{l'} \widetilde{H}^\alpha_{l'l} u_l, \tag{37}$$

where we have further defined the $m \times m$ matrix

$$\widetilde{H}^\alpha = \begin{pmatrix} A^\alpha & (\widetilde{B}^\alpha)^T \\ \widetilde{B}^\alpha & \hat{A}^\alpha \end{pmatrix}. \tag{38}$$

Here $\widetilde{B}^\alpha$ is obtained from B^α by cancelling the j_0-th row, and $\hat{A}^\alpha$ from A^α by cancelling the j_0-th row and column. Thus $\widetilde{B}^\alpha$ is an $(n-1) \times n$ matrix and $\hat{A}^\alpha$ an $(n-1) \times (n-1)$ matrix.

Application of the above criterion means now the following. For a given set of m observables one first has to construct the $m \times m$ matrices $\widetilde{H}^\alpha$, and then one builds from their columns according to the sets $\{k_1, \ldots, k_m\}$ the matrices

$$\widetilde{W}(k_1, \ldots, k_m) = \begin{pmatrix} \widetilde{H}^1_{1k_1} & \cdots & \widetilde{H}^m_{1k_m} \\ \vdots & & \vdots \\ \widetilde{H}^1_{mk_1} & \cdots & \widetilde{H}^m_{mk_m} \end{pmatrix}. \tag{39}$$

If at least one of the determinants of $\widetilde{W}(k_1, \ldots, k_m)$ is nonvanishing then one has a complete set, up to the already mentioned discrete ambiguities.

We will illustrate this for the above considered simplest nontrivial case $n = 2$ choosing $\mathcal{O}_0$, $\mathcal{O}_1$, and $\mathcal{O}_2$ for which one has with $j_0 = 1$

$$\widetilde{\sigma}_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \widetilde{\sigma}_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \widetilde{\sigma}_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \tag{40}$$

One finds in this case three $\widetilde{W}$ matrices which possess a nonvanishing determinant

$$\widetilde{W}(1,1,1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \widetilde{W}(2,2,1) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \widetilde{W}(3,1,3) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \tag{41}$$

Thus, the three observables constitute a complete set. For this case, any group of three observables is complete and the direct solution is simpler. The usefulness of this new criterion becomes apparent if one considers reactions with a larger number of independent T-matrix elements, as is done in the next section.

V. APPLICATION TO THE LONGITUDINAL OBSERVABLES IN ELECTROMAGNETIC DEUTERON BREAK UP

Now we will consider the more complex case of deuteron electrodisintegration. An extensive formal discussion of polarization observables may be found in [6,8]. The total number of T-matrix elements is $3 \times 3 \times 4 = 36$ which is reduced by parity conservation to $n = 18$. Of these, 6 are associated with the charge or longitudinal current density component while the remaining 12 belong to the transverse current density components. Only the latter appear in photodisintegration. In order to simplify the present discussion, we will restrict ourselves to the purely longitudinal contribution.

We would like to remind the reader that the differential cross section and the polarization components of the outgoing particles are completely described by the so-called structure functions, which can be separated experimentally by appropriate kinematic settings as is discussed in detail in [11]. For the present discussion, only the longitudinal

ones will be considered. In the following, we will use the term "observable" for a longitudinal structure function $f_L^{(')IM}(X)$, where $X = 1$ refers to the differential cross section, $X = x_j$, y_j, and z_j to the respective polarization components $P_x(j)$, $P_y(j)$, and $P_z(j)$ of one outgoing nucleon ($j = 1, 2$), and $X = x_i x_j$ to the polarization components $P_{x_i} P_{x_j}$ of both outgoing nucleons. Since one has six independent longitudinal T-matrix elements, each observable is represented by a hermitean 6×6 matrix whose explicit form depends on the adopted basis for the T-matrix and the chosen reference frame.

We will use here the same definition of the polarization components as in [8] which is based on the Madison convention. Our standard form of the T-matrix is characterized by the quantum numbers $\{s, m_s, \lambda, m_d\}$ of the total spin of the two outgoing nucleons, its projection on their relative momentum, the spin projections of the virtual photon and the deuteron on the momentum transfer, respectively. With respect to the definition of the observables in terms of the T-matrix elements, we refer to [8] where a complete listing of all possible observables is given. However, for the present discussion the representation of the T-matrix in the helicity basis is more advantageous, because the hermitean matrices of the observables have a simpler structure. In this case, the quantum numbers of the T-matrix elements are given by the spin projections of the participating particles on the respective momenta $\{\lambda_p, \lambda_n, \lambda, \lambda_d\}$. We have numbered the independent matrix elements according to the listing in Table I.

For the longitudinal observables, one finds a total number of 72 whereas the total number of linearly independent observables is 36. It is obvious that the observables of a complete set should be selected from linearly independent ones. A possible choice for a set of linearly independent observables is listed in Table II, and we will restrict the following discussion to this set. However, any other set of linearly independent observables, as may be obtained from the total number of 72 by taking into account the linear relations of [8], can also be used. For the observables of type A in Table II, one has five IM-values and for type B four (see [8]).

Now a complete set of longitudinal observables in the sense described in the introduction should contain 11 observables only, but the question is which ones. In order to select a proper subset of 11 from a set of 36 linearly independent observables, we have now applied our new criterion by first constructing the corresponding matrices $\widetilde{H}^\alpha$ and then by checking the determinants of the matrices $\widetilde{W}(k_1, \ldots, k_{11})$ for various choices of $(k_1, \ldots, k_{11})$. First, we did a restricted search by setting $j_0 = 1$ and $k_\alpha = 1$, ($\alpha = 1, \ldots, 11$), and have found as possible complete sets of observables those listed in Table III. Here and in the following tables the superscripts refer to the IM-values of the corresponding observables. The meaning of this table is that from each of the 11 columns one may pick arbitrarily one of the listed observables in order to form a complete set. Other possible complete sets, which are obtained by setting $j_0 = 3$ and $k_\alpha = 3$, ($\alpha = 1, \ldots, 11$), are listed in Table IV. Finally, allowing for more general choices for $(k_1, \ldots, k_{11})$, we found that there is only a very weak restriction on the possible sets. In fact, in order to obtain a complete set one may select from the total 36 observables almost any set of 11 observables. The only limitation is that the set should not include more than eight of the 12 observables which are listed in Table V, since otherwise the set would be incomplete. This restriction is a consequence of a specific structure of the matrices A^α and B^α for the listed observables. We note in passing, that all these complete sets fulfil the necessity criterion of Simonius [9].

Any complete set of 11 observables leads to a system of 11 nonlinear coupled equations for the T-matrix elements. For the sets listed in Table VI we solved the resulting equations numerically by using as input the observables resulting from a theoretical calculation as presented in [10,11], where we have chosen arbitrarily a specific internal excitation energy ($E_{np} = 100$ MeV) and momentum transfer ($q_{c.m.}^2 = 5$ fm^{-2}) for various n-p angles θ_{np}. The first two sets are cases from Tables III and IV, while the two last sets do not correspond to the application of our criterion with a specific choice of a column $k_\alpha = k_0$. In general, we found a larger number of solutions which is a manifestation of the previously mentioned discrete ambiguities. On the other hand, we already knew the correct answer from the knowledge of the theoretical T-matrix which served as input for the evaluation of the observables. Thus it was easy to check whether the correct solution was among the ones found numerically, as was the case. However, it is worth mentioning that it was not always easy to find the proper solution. While for the sets 1, 2, and 4 it was not difficult to obtain the correct solution among the various possible ones, it was extremely tedious to locate it for set 3.

In an experimental study, however, the correct solution is not known in advance. Therefore, in order to decide, which of the various solutions found in general is correct, one has to calculate with the obtained solutions for the T-matrix elements additional observables and compare them to their measured values. Thus for any complete set one needs to measure further observables in order to be able to single out a unique solution for the T-matrix. For example, for our first set of Table VI it was sufficient to consider one more observable, namely x_2^{10}. But in general it might be necessary, to check several additional observables. A further discussion of electromagnetic deuteron break-up with regard to complete sets of observables including transverse and interference observables will be presented in a forthcoming paper.

We hope that this nontrivial example demonstrates convincingly the usefulness of our criterion for selecting a complete set of observables for a given reaction and that it will provide a guideline for polarization studies in the future.

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TABLE I. Numbering of independent longitudinal T-matrix elements ($\lambda = 0$) of $d(e, e'N)N$ for the helicity basis.

j	1	2	3	4	5	6
λ_p	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
λ_n	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$
λ_d	0	0	-1	-1	1	1

TABLE II. A set of linearly independent longitudinal structure functions $f_L^{IM}(X)$ of $d(e, e'N)N$ with IM-values $(00, 11, 20, 21, 22)$ for the A-type and $(10, 11, 21, 22)$ for the B-type observables.

X	1	xx	xz	y_1	x_1	x_2	z_1	z_2
type	A	A	A	A	B	B	B	B

TABLE III. Complete sets of longitudinal observables of $d(e, e'N)N$ by setting $j_0 = 1$.

1^{00}	y_1^{11}	1^{21}	xz^{00}	x_1^{11}	z_1^{11}	y_1^{00}	1^{11}	y_1^{21}	x_1^{21}	z_1^{21}
1^{20}	xz^{21}	xx^{21}	xz^{20}	x_2^{11}	z_2^{11}	y_1^{20}	xx^{11}	xz^{11}	x_2^{21}	z_2^{21}
xx^{00}										
xx^{20}										

TABLE IV. Alternative complete sets of longitudinal observables of $d(e,e'N)N$ by setting $j_0 = 3$.

1^{00}	x_1^{11}	1^{21}	x_1^{10}	1^{22}	x_2^{10}	y_1^{00}	1^{11}	x_1^{21}	x_1^{22}	z_1^{22}
1^{20}	x_2^{11}	z_1^{11}	xz^{00}	xx^{00}	xz^{22}	y_1^{20}	z_1^{21}	x_2^{21}	y_1^{22}	z_2^{22}
z_1^{10}	y_1^{11}	z_2^{11}	xz^{20}	xx^{20}		y_2^{22}	z_2^{21}	y_1^{21}		
z_2^{10}	xz^{21}	xx^{21}					xx^{11}	xz^{11}		
xx^{22}										

TABLE V. Listing of longitudinal observables of $d(e,e'N)N$ which restrict the choice of complete sets. An inclusion of more than eight of the listed observables into a set of 11 observables would result automatically in an incomplete set.

1^{22}	x_1^{10}	x_2^{10}	x_1^{22}	x_2^{22}	y_1^{22}	z_1^{10}	z_2^{10}	z_1^{22}	z_2^{22}	xx^{22}	xz^{22}

TABLE VI. Selected complete sets for a numerical solution

1^{00}	1^{11}	1^{21}	y_1^{00}	y_1^{11}	y_1^{21}	x_1^{11}	x_1^{21}	z_1^{11}	z_1^{21}	xz^{00}
1^{00}	1^{11}	1^{21}	1^{22}	y_1^{00}	y_1^{11}	y_1^{21}	y_1^{22}	x_1^{10}	x_2^{10}	z_1^{22}
1^{00}	1^{11}	1^{20}	1^{21}	1^{22}	y_1^{00}	y_1^{11}	y_1^{20}	y_1^{21}	y_1^{22}	x_1^{10}
1^{00}	1^{11}	1^{20}	1^{21}	1^{22}	y_1^{00}	y_1^{11}	y_1^{21}	y_1^{22}	x_1^{10}	z_1^{10}